6-90-62-116 • FEBRUARY 1963

403 797

TECHNICAL REPORT: COMMUNICATIONS

DIFFRACTION PATTERN ANALYSIS OF A RECTANGULAR APERTURE IN THE PRESENCE OF ABERRATIONS

PART 1

DERIVATION OF A GENERAL SOLUTION

NOTICE

WORK CARRIED OUT AS PART OF THE LOCKHEED INDEPENDENT RESEARCH PROGRAM.

6-90-62-116 ● FEBRUARY 1963

6-90-62-116

TECHNICAL REPORT: COMMUNICATIONS

DIFFRACTION PATTERN ANALYSIS OF A RECTANGULAR APERTURE IN THE PRESENCE OF ABERRATIONS

PART 1

DERIVATION OF A GENERAL SOLUTION

by
HERMANN P. GREINEL

WORK CARRIED OUT AS PART OF THE LOCKHEED INDEPENDENT RESEARCH PROGRAM

Lockheed

MISSILES & SPACE COMPANY

A GROUP DIVISION OF LOCKHEED AIRCRAFT CORPORATION

SUNNYVALE, CALIFORNIA

NOTICE

FOREWORD

This report was prepared at the Electronic Sciences Laboratory as part of the Independent Research program of LMSC. It is the first of a series of contributions toward finding appropriate diffraction-pattern manipulation to improve the discrimination probability of aberrant optical systems for off-axis objects.

ABSTRACT

This paper discusses a general solution of the diffraction integral for a rectangular aperture considered in scanning procedures of diffraction patterns. It is shown that in the presence of aberrations, the integrand in Fresnel-Kirchhoff's diffraction formula can be expanded into a series. By this expansion, the two-dimensional diffraction integral is converted into a summation, the single terms of which divide into products of two functions where only one-dimensional integrations over products of Legendre polynomials are to be performed. In these integrations, use is made of the orthogonality property of Legendre's polynomials.

Particular solutions are obtained concerning the two-dimensional diffraction pattern intensity distribution in the Gaussian image plane for an aberrant optical system and the three-dimensional intensity distribution near the focus of an aberration-free optical system.

CONTENTS

LIST OF SYMBOLS

1. Fundamental symbols

P_o	point in the object space
P_1^*	geometrical image point of P_o
P	typical point in the neighborhood of P_1^*
P_1'	arbitrary point in the plane of the exit pupil
C	center of the exit pupil
R	radius of the Gaussian reference sphere through C with center at P_1^*
Q	intersection point of the ray $[P_1' P]$ with the reference sphere
$\bar{Q}$	intersection point of the ray $[P_1' P]$ with the actual wavefront passing through C
s	distance between Q and P
$\Phi = [\bar{Q} Q]$	wave aberration
$k = 2\pi/\lambda$	wave number
λ	wavelength of at least quasi-monochromatic light
dS	surface element on the reference sphere
A	amplitude of disturbance function
χ	angle between diffracted ray and normal to the sphere at Q
$U(P)$	value of the disturbance function at point P
$2a$, $2b$	linear dimensions of rectangular aperture acting as entrance pupil
M	Gaussian lateral magnification factor between the object and image planes

M'	Gaussian lateral magnification factor between both pupil planes
X'_1, Y'_1, Z'_1	Cartesian coordinates of P'_1 referred to system with origin at P^*_1
x, y, z	Cartesian coordinates of Q referred to system with origin at C
X', Y', Z'	Cartesian coordinates of Q referred to system with origin at P^*_1
$\bar{X}', \bar{Y}', \bar{Z}'$	Cartesian coordinates of $\bar{Q}$ referred to system with origin at P^*_1
X, Y, Z	Cartesian coordinates of P referred to system with origin at P^*_1; these coordinate systems have mutually parallel and equally oriented axes with common Z-axis along the line $[CP^*_1]$
$\mu_o = \dfrac{M'a}{R}$, $\nu_o = \dfrac{M'b}{R}$	dimensionless width and length of rectangular area in the exit pupil, respectively, corresponding to aperture
$\xi = \dfrac{x}{R\mu_o}$, $\eta = \dfrac{y}{R\mu_o}$	dimensionless coordinates of point Q on reference sphere
$u = kZ$, $v = kX$, $w = kY$	dimensionless "optical coordinates" of observation point P
X_o, Y_o	Cartesian coordinates of P_o referred to system with origin at the center of the object plane
X^*_1, Y^*_1	Cartesian coordinates of P^*_1 referred to system with origin at the center of the image plane; arbitrarily choosing the meridional plane defined by optical axis and principal ray as reference plane, one gets $X_o = X^*_1 \equiv 0$
X'_o, Y'_o, Z'_o	Cartesian coordinates at point P'_o where a specific ray emerging from P_o intersects the aperture plane; reference system has origin at the object plane center

X_1', Y_1', Z_1'	Cartesian coordinates of point P_1' in the exit pupil plane which corresponds to P_o' (with or without diffraction); reference system has origin at the image plane center
X_1, Y_1	Cartesian coordinates of point P_1 in the image plane, aberrant by small amounts from X_1^* and Y_1^*
x_1, y_1, z_1	Cartesian coordinates of observation point P in the neighborhood of P_1^*, referred to system with origin at the image plane center
X', Y', Z'	Cartesian coordinates of intersection points Q_1 and Q of rays $[P_1' P_1]$ and $[P_1' P]$, respectively, with Gaussian reference sphere centered at P_1^* and passing through the center C of exit pupil
$\bar{X}', \bar{Y}', \bar{Z}'$	Cartesian coordinates of intersection points $\bar{Q}_1$ and $\bar{Q}$ of rays $[P_1' P_1]$ and $[P_1' P]$, respectively, with actual wavefront through C_1; the coordinate systems used here have also mutually parallel and equally oriented axes with common Z-axis along the optical axis of the image forming instrument (positive in the sense of light-wave propagation from the object to the image space)

$$\left.\begin{array}{ll} X_o = \dfrac{D_o x_o}{n_o \lambda_o} \; ; & Y_o = \dfrac{D_o y_o}{n_o \lambda_o} \\[2ex] X_1 = -\dfrac{D_1 x_1}{n_1 \lambda_1} \; ; & Y_1 = -\dfrac{D_1 y_1}{n_1 \lambda_1} \\[2ex] X_o' = \lambda_o \xi_o \; ; & Y_o' = \lambda_o \eta_o \\[1ex] X_1' = \lambda_1 \xi_1 \; ; & Y_1' = \lambda_1 \eta_1 \end{array}\right\} \text{Seidel variables}$$

n_o, n_1	refractive indices of object and image spaces, respectively, without restricting the generality one may use $n_o = n_1 = 1$
D_o	distance between object and entrance pupil planes
D_1	distance between image and exit pupil planes; for convenience, $D_1 \approx R$ is used where R is the radius of the Gaussian reference sphere

λ_o, λ_1	unit lengths in the entrance and exit pupil planes, respectively, λ_o is arbitrarily chosen equal to unity
$x_o = x_1^*$; $y_o = y_1^*$	by Gaussian optics, referred to Seidel's coordinates of P_o and P_1^*
$r_o^2 = x_1^{*2} + y_1^{*2}$ $\rho^2 = \xi_1^2 + \eta_1^2$ $\kappa^2 = x_1^* \xi_1 + y_1^* \eta_1$	basic quantities needed for analytical description of wave aberration in Seidel coordinates
$x_1^* = -\lambda_1 X_1^*/R$; $y_1^* = -\lambda_1 Y_1^*/R$ $\xi_1 \approx x/\lambda_1 = R\mu_o \xi/\lambda_1$; $\eta_1 \approx y/\lambda_1 = R\nu_o \eta/\lambda_1$	relation between Seidel's coordinates and integration variables used
$\Phi^{(2k)}$	aberration function of order $(2k - 1)$
$\Phi^{(4)}$	primary or third-order wave aberration
B	coefficient of third-order spherical aberration
C	coefficient of third-order astigmatism
D	coefficient of third-order curvature of field
E	coefficient of third-order distortion
F	coefficient of third-order coma

2. Symbols used in expansions and evaluation

$V_n^m(\rho \sin \theta, \rho \cos \theta)$ $= R_n^m(\rho)\, e^{jm\theta}$	Zernike's circle polynomials in two real variables $x = \rho \sin \theta$; $y = \rho \cos \theta$

$R_n^m(\rho)$	radial polynomials, n , m being arbitrary indices restricted by $m \gtreqless 0$ and $n \geqq 0$ integer, $n \geqq \lvert m \rvert$, $n - \lvert m \rvert$ even
$\left.\begin{array}{l}\delta_\alpha^\beta = 1 \text{ for } \alpha = \beta \\ \quad = 0 \text{ for } \alpha \neq \beta\end{array}\right\}$	Kronecker symbol referred to arbitrary indices α , β
$J_{s+1/2}(\rho)$	Bessel function of order $(s + 1/2)$, s being integer, with arbitary argument ρ
$P_s(\cos\theta)$	Legendre polynomial of order s in usual definition
$\beta_{n,\nu}/\delta'_n$	coefficient in defining the Legendre polynomial of order n by a power series; δ'_n is common denominator for all values $\beta_{n,\nu}$, ν is running index
$\alpha_{n,\nu}/\delta_n$	coefficient in expanding the n^{th} power in terms of Legendre's polynomials, δ_n being again a common denominator for all values $\alpha_{n,\nu}$
$K^\ell_{n-m,m}$	constant in expressing the ℓ^{th} power of the wave aberration Φ as a double series in terms of power products $(\xi^{n-m}\eta^m)$ where ξ , η are integration variables
ℓ	index in $K^\ell_{n-m,m}$ like $(n - m)$ and n , as well as running quantity in power series expansion of exponential function $e^{jk\Phi}$
$V_\ell^{n-m}(u, v)$, $W_\ell^m(u, w)$	solution functions occurring in the expanded form of the disturbance $U(u, v, w)$ at the observation point $P(u, v, w)$; (conversion of two-dimensional diffraction integral into a series containing only one-dimensional integrals)
$\mathcal{P}_r^k(x) = x^k P_r(x^2)$	new introduced polynomials expressible as a double series of Legendre polynomials, or even as a single series of the form $$\mathcal{P}_r^k(x) = \sum_{i=o}^{\frac{2r+k-(1)}{2}} \frac{N^k_{r,i}}{D^k_r} P_{2r+k-2i}(x)$$

	where x is an arbitrary variable, r, k, i are subscripts; D_r^k is common denominator for all coefficients $N_{r,i}^k$
$G_r^k(z)$	newly defined solution function of arbitary variable z, obtained by applying orthogonality property of Legendre polynomials to $$\int_{-1}^{1} \wp_r^k(x)\, P_{r'}(x)\, dx$$
$I(u, v, w) = \|U(u, v, w)\|^2$	intensity distribution of diffraction pattern at $P(u, v, w)$
$I_o[(A/\lambda)4\mu_o\nu_o]^2$	central (maximum) intensity of aberration-free diffraction pattern

Section 1
INTRODUCTION

Determination of the intensity distribution of a diffraction pattern is based on the evaluation of the Fresnel-Kirchhoff diffraction formula (Ref. 1). Although this formula is obtained only as an approximation from the more general Helmholtz-Kirchhoff integral theorem, its usefulness and validity have been proven in many cases in the past.

Historically, the classical evaluation technique of the diffraction integral yielded the discrimination between Fresnel and Fraunhofer diffraction patterns which, of course, have been widely investigated under a variety of conditions for the parameters involved. Relatively soon after the publication of the Fresnel-Kirchhoff formula in 1882, Lommel (Ref. 2) extended the evaluation technique in order to determine the three-dimensional intensity distribution of the diffraction patterns in the neighborhood of the Gaussian focus of an aberration-free optical system due to circular and rectangular apertures. Later, attempts were made to study the effect of aberrations on diffraction patterns, especially with regard to circular apertures. As pointed out by Wolf (Ref. 3) in a historical review, these researches, however, cannot be considered as entirely satisfactory because they are either too restricted or involve heavy computations. In his important thesis of 1942, Nijboer (Ref. 4) obtained a simpler and more satisfactory solution of the problem for cases where, for a circular aperture, the wave deformation produced by the geometrical optical aberrations is only a fraction of a wavelength. Instead of using Lommel's series expansions in terms of Bessel functions, Nijboer applies expansions of the aberration function in terms of the circular polynomials introduced by Zernike (Ref. 5). He obtains a formally different but equivalent representation to that of Lommel for the three-dimensional intensity distribution near the focus of an aberration-free optical system. However, by this procedure, he is also able to incorporate the effects of the aberrations.

The literature on diffraction by rectangular apertures in the presence of aberrations is rather sparse. On the other hand, because of their specific structure, diffraction patterns of rectangular apertures are believed more advantageous than those of circular or annular apertures in cases where the geometrical image location of an off-axis object is to be determined by a two-dimensional scanning procedure. For this reason, it is attempted in this paper to obtain a new solution of the Fresnel-Kirchhoff diffraction integral for a stationary rectangular aperture. This solution represents the pattern near the focus of the optical system in the presence of geometrical optical aberrations due to a point-source type object at infinite distance in an off-axis position.

Section 2
ASSUMPTIONS AND RESTRICTIONS IN APPLYING THE FRESNEL-KIRCHHOFF FORMULA TO DIFFRACTION PROBLEMS

It has been pointed out earlier that the Fresnel-Kirchhoff formula can be considered only as an approximation. The reason for this can be seen in the fact that diffraction theory cannot be treated in a completely general manner, i.e., it cannot be rigorously regarded as describing the diffraction in telescopic optical systems under all conditions. Therefore, it is deemed advisable to describe, in this paper, the assumptions and restrictions to which Fresnel-Kirchhoff's diffraction integral is subjected.

Although known for centuries (Leonardo da Vinci first mentioned the appearance of diffraction phenomena), diffraction could not be explained on the basis of the corpuscular theory of light until the early nineteenth century. Fresnel, in 1816, first showed from wave theory considerations that diffraction phenomena can be explained by the application of both the Huygens' principle of secondary wavelets and Young's interference principle. Today, the treatment of diffraction problems is based on the concept of the electromagnetic wave nature of light, and consequently Maxwell's equations are employed as the fundamental basis for this treatment.

In its usual formulation, diffraction theory presumes that a scalar wave equation can be applied in solving the problem of describing the vector quantities of the electromagnetic field. From this point of view, first of all, it has to be shown that the use of a single scalar wave instead of a vectorial wave is justified in treating diffraction problems of instrumental optics. An example of this justification may be found in Born-Wolf's Principles of Optics, p. 386 (Ref. 1).

Mathematically, then, the inhomogeneous scalar wave equation, representing a significant simplification of the problem to be solved, is separated in a well-known

manner into two equations by assuming the general solution in the form of a product of two functions, one of which is time dependent and the other space dependent. The space-dependent portion of the solution, satisfying Helmholtz's homogeneous wave equation, enters the Helmholtz-Kirchhoff integral theorem which expresses this solution at any point in the field in terms of the values of the solution and its first derivative along the inward normal at all points on an arbitrary closed surface surrounding the observation point. It is to be noticed that in order to obtain the solution, the determination of the appropriate Green's function is possible only in the simplest cases, i.e., where the surrounding surface is a plane.

Kirchhoff's approximation, derived from the more general integral theorem, consists of a reduction of the problem to a much simpler form.

First, the closed surface around the observation point is assumed to be formed by three parts:

- A sharp-edged opening in a plane screen through which light from the object point outside the surface penetrates to the observation point. If there is no other obstacle between object point and diffracting aperture, the wavefront of the incident light, of course, is spherically shaped
- A portion of the nonilluminated side of the screen
- A portion of a sphere of sufficiently large radius with center at the observation point

It is furthermore assumed that the linear dimensions of the diffracting aperture are large compared with the wavelength of the monochromatic (or at least quasi-monochromatic) light emerging from the source outside the closed surface and small compared with the distances of the object and observation points from the plane screen.

Next, it is supposed that the disturbance function and its first derivative along the inward surface normal at any point of the opening, except points in the immediate neighborhood of the aperture rim, will not differ appreciably from the values of the

same quantities encountered in the absence of the screen. In evaluating the diffraction integral, the contributions from these quantities encountered at excepted points are assumed to be negligibly small. It is also supposed that contributions will be approximately zero from points on the nonilluminated side of the screen. In mathematical formulation these assumptions constitute Kirchhoff's boundary conditions. The additional, physically reasonable assumption of not strictly monochromatic waves, then, makes the integral over the spherical portion of the closed surface vanish as the radius of the sphere goes to infinity (see Born-Wolf's textbook (Ref. 1) for details). Thus, the only remaining portion of the closed surface contributing to the disturbance at the observation point is the aperture itself.

The introduction of Kirchhoff's boundary conditions has proved satisfactory in many cases. As a theory, however, it does not indicate the cases for which the conditions are no longer applicable.

Evidently, the aperture in the plane screen may be replaced by an open portion of any other surface as long as the rim of the new opening represents the intersection of the new surface with the boundary surface of the pencil of rays defined by the object point and the plane aperture. Of course, the closed surface around the observation point, now, has to be completed by the portion of this boundary surface between the edges of the plane aperture and the rim of the new opening. The disturbance at the observation point is not affected if the contributions to the integral from that boundary surface portion are negligibly small. This assumption will certainly hold as long as the radius of curvature at any point of the new opening is sufficiently large, which is fulfilled if one chooses an arbitrary, in particular a spherical, wavefront as the new intersecting surface.

These considerations substantially simplify the general Fresnel-Kirchhoff diffraction integral. Its evaluation, then, yields automatically the discrimination between Fresnel and Fraunhofer diffraction. The simpler case of Fraunhofer diffraction, involving plane waves emerging from a point source at infinity and diffracted into an observation

plane at infinity, is much more important in practical optics than that of Fresnel diffraction, which primarily deals with spherical waves. Unfortunately, infinite distances of object and observation points from the screen are not realizable in practice. For simulating the ideal case, one has to use a collimating optical system between the radiation source (of finite extension) and the diffracting aperture in order to generate an approximately plane wave front, as well as an imaging system behind the screen in order to produce a converging beam of light with which to observe the diffraction pattern at finite distance from the screen. Nevertheless, the fact that neither ideal point sources, even without using a collimating system, nor sharp-edged apertures are realizable in practice, has to be considered when applying the Fresnel-Kirchhoff diffraction integral insofar as boundary effects are concerned.

As long as one can neglect the restrictions and one stays in the validity range of geometrical optics, one obtains, in the Gaussian focal plane, as the image of an arbitrary point source, the characteristic diffraction pattern as described by the Fresnel-Kirchhoff integral in the absence of aberrations.

If one considers the three-dimensional intensity distribution of the diffraction pattern near the focus, one must extend the integrand in Fresnel-Kirchhoff's integral (as done, for example, by Lommel), even in the case of an aberration-free optical system.

As one exceeds the limits of Gaussian optics, the wave fronts in the converging beam behind the aperture deviate more and more from their spherical shape because of the geometrical optical aberrations. Consequently, considerable image deformations are to be observed. If the aberrations are large enough, the defect of the optical images obtainable, in general, is studied by geometrical optics. If, however, as in the case considered in this paper, the aberrations are sufficiently small, the image deformation has to be treated by the diffraction theory of aberrations.

Section 3
THE DIFFRACTION INTEGRAL FOR A RECTANGULAR APERTURE

The Fresnel-Kirchhoff diffraction integral, in the specific case treated in this paper, is to be used in considering the disturbance $U(P)$ at a typical point P in the neighborhood of the geometrical image point, P_1^*, of a point P_o in the object space. It is derived from the geometrical configuration encountered in Fig. 1.

The planes that are perpendicular to the optical axis of the image-forming instrument and contain the points P_o or P_1^* are usually denoted as object or image planes, respectively. The lateral magnification factor M between these planes is determined by the classical rules of geometrical optics. The plane rectangular aperture acts as the entrance pupil of the optical system. Application of the same classical rules yields the exit pupil location and the Gaussian lateral magnification factor M' between both pupil planes which are also considered to be perpendicular to the optical axis. The meridional plane, defined by P_o (and P_1^*) and the optical axis, is taken as a reference plane.

Two Cartesian coordinate systems are introduced with origins at C and P_1^*, respectively. The positive Z-axes have the direction of the ray from C to P_1^* (direction of light propagation); the other axes are mutually parallel and have equal positive directions.

For convenience, the Y-axes are chosen in the above-defined meridional plane. Points in the system with origin at C are described by coordinates x, y, z, and points in the other system by X, Y, Z.

The aperture (in the exit pupil plane) is now replaced by that portion of the Gaussian reference sphere centered at P_1^* which passes through C and approximately fills

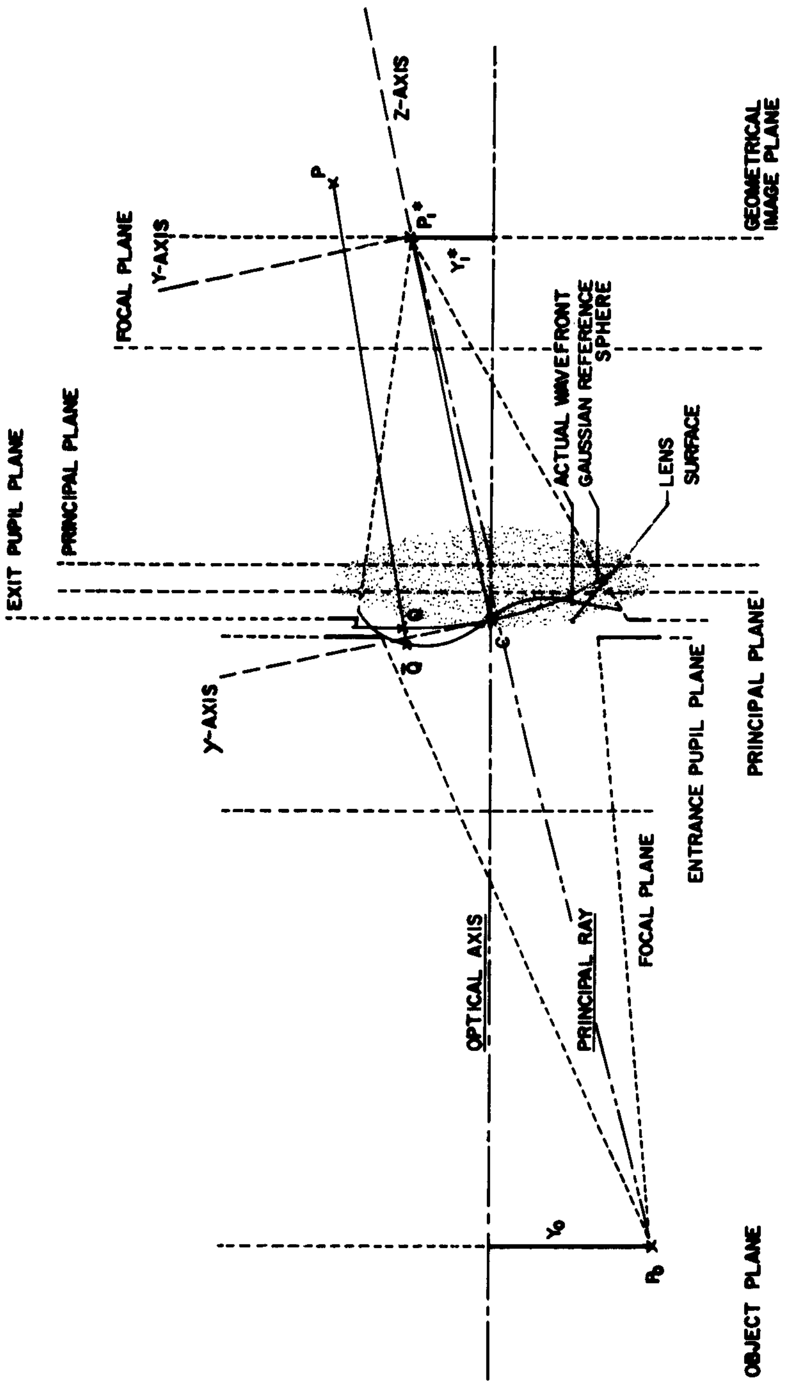

Fig. 1 Situation Encountered in Treating the Diffraction Problem

this aperture. The actual wave front passing through C, in general, will deviate slightly from the spherical shape because of the small aberrations being assumed. An arbitrary ray diffracted into the image space in the direction of the observation point $P(X, Y, Z)$ in the neighborhood of P_1^* passes the aperture at $P_1'(X_1', Y_1', Z_1')$ and intersects the Gaussian reference sphere and the actual wave front at $Q(X', Y', Z')$ and $\bar{Q}(\bar{X}', \bar{Y}', \bar{Z}')$. It also is possible to describe Q by its coordinates x, y, z. Of course, these coordinates may be deduced from the other ones by the transformation

$$X' = x; \; Y' = y; \; Z' = z - R$$

Because P is a point in the immediate neighborhood of P_1^* and the aperture's linear dimensions are assumed to be small compared to the radius R of the Gaussian reference sphere, the angle χ between the diffracted ray and the normal to the sphere at Q will be small. Thus, it can be assumed that the variation of the inclination factor in Fresnel-Kirchhoff's general diffraction integral will become small as Q explores the portion of the reference sphere corresponding to the aperture area. For this reason, this variation may be neglected.

Then, denoting the distance between a typical point Q on the reference sphere and the observation point, P, by s, and remembering that the distance between $\bar{Q}$ and Q defines the wave aberration Φ, the diffraction integral takes the form

$$U(P) = -j \frac{A}{\lambda} \iint \frac{\exp\left[jk(\Phi + s - R)\right]}{sR} \, dS$$

where:

$k = 2\pi/\lambda$	=	the wave number
λ	=	the wavelength of the monochromatic or quasi-monochromatic light
A	=	the amplitude of the wave (assumed to be substantially constant over the wave front)
dS	=	the surface element on that portion of the reference sphere over which the integration is to be extended

To obtain an approximate solution of the integral, for observations points P in the region of the geometrical image point P_1^*, the quantity s in the denominator of the integrand usually is replaced by the radius R of the reference sphere. The surface element is taken as

$$dS = dx\,dy$$

Taking the integration limits as $\pm M'a$ and $\pm M'b$, where 2a and 2b are the linear dimensions of the entrance pupil (diffracting aperture), is restricted to cases where only relatively small incidence angles of the principal ray are involved (supposition of small aberrations).

From the analytical expressions for the distance s, the radius R of the reference sphere and the distance r between P_1^* and P, supposed to be small, one obtains

$$s - R \approx Z - \left(\frac{Xx + Yy}{R} + \frac{Z}{2} \frac{x^2 + y^2}{R^2} \right)$$

In this expression, all small terms of second order have been neglected.

Hence

$$U(X, Y, Z) = -j \frac{A}{\lambda} e^{jkZ} \int_{-M'a}^{M'a} \int_{-M'b}^{M'b} \exp\left[jk \left(\Phi - \frac{Xx + Yy}{R} - \frac{Z}{2} \frac{x^2 + y^2}{R^2} \right) \right] \frac{1}{R^2}\, dx\,dy$$

For convenience, new integration variables are introduced, and new parameters describing the observation point location in the geometrical image region are defined.

Integration variables:

$$x = R\mu = R\mu_o \xi \qquad\qquad y = R\nu = R\nu_o \eta$$

$$dx = R\,d\mu = R\mu_o\,d\xi \qquad\qquad dy = R\,d\nu = R\nu_o\,d\eta$$

Integration limits:

$$x = \pm M'a \qquad\qquad y = \pm M'b$$

$$\mu = \pm M'a/R = \pm\mu_o \qquad\qquad \nu = \pm M'b/R = \pm\nu_o$$

$$\xi = \pm 1 \qquad\qquad \eta = \pm 1$$

Optical coordinates:

$$u = kZ \qquad\qquad v = kX \qquad\qquad w = kY$$

Hence

$$U(u,v,w) = -j\frac{A}{\lambda}4\mu_o\nu_o e^{ju}\frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}\exp\left[j\left(k\Phi - \frac{u}{2}\mu_o^2\xi^2 - v\mu_o\xi - \frac{u}{2}\nu_o^2\eta^2 - w\nu_o\eta\right)\right]d\xi\,d\eta$$

In the case of an aberration-free optics, i.e., for $\Phi = 0$, obviously the two-dimensional diffraction integral is expressible as the product of two one-dimensional integrals. It seems appropriate at this point to attempt an adequate separation also for the optical system affected by aberrations.

Usually, to obtain an analytical expression for the wave aberration, two mutually parallel and equally oriented Cartesian coordinate systems are introduced with origins on the optical axis of the image-forming instrument at the centers of the object and image planes, respectively. The positive Z-axes are taken along the

optical axis in direction from the object to the image space. The object and image points are defined by $P_o(X_o, Y_o, 0)$ and $P_1^*(X_1^*, Y_1^*, 0)$, respectively. If M is the Gaussian lateral magnification factor between the object and image planes, one has

$$X_1^* = MX_o \qquad \text{and} \qquad Y_1^* = MY_o$$

(Arbitrarily placing the Y-axes into the meridional reference plane, defined in the beginning, one will obtain $X_o = X_1^* \equiv 0$.)

The center of the exit pupil plane is denoted by $C(0, 0, -D_1)$ where D_1 is the distance between the exit pupil and image planes. Again, the Gaussian reference sphere is centered at P_1^* and passed through C. The actual wave front also passes through C. If, for the moment, diffraction effects are omitted, a specific nondiffracted ray emerging from P_o will meet the entrance pupil at $P_o'(X_o', Y_o', Z_o')$, the exit pupil at $P_1'(X_1', Y_1', Z_1')$, and the image plane at $P_1(X_1, Y_1, 0)$, (see Fig. 2).

If the ray $[P_1' P_1]$ intersects the actual wave front and the Gaussian reference sphere at $\bar{Q}_1(\bar{X}', \bar{Y}', \bar{Z}')$ and $Q_1(X', Y', Z')$, respectively, the aberration function is defined as the distance between these two points

$$\Phi = [\bar{Q}_1 Q_1]$$

By making use of the fact that both $\bar{Q}_1$ and C are located on the same wave front one obtains

$$\Phi = [P_o Q_1] - [P_o C]$$

i.e., Φ is expressible as the difference of two light paths. Because Q lies on the Gaussian reference sphere, its coordinate Z can be eliminated if both optical paths are expressed in the notations of the point characteristic, and one obtains finally

$$\Phi = \Phi(X_o, Y_o; X', Y) \qquad \text{(Ref. 1, p. 202)}$$

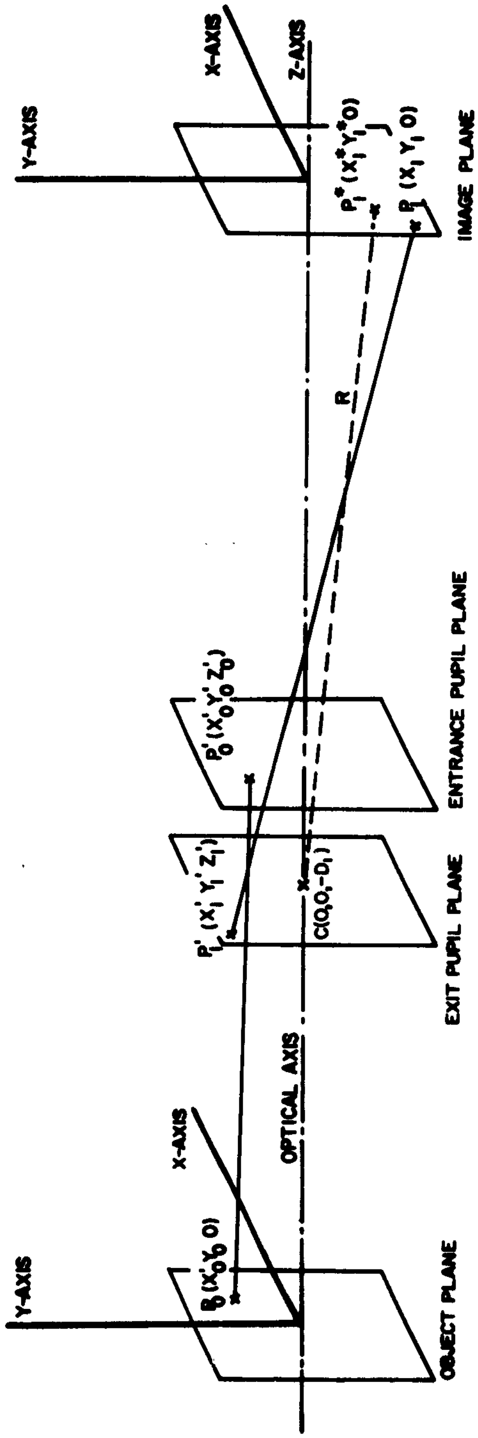

Fig. 2 Situation Encountered in Handling the Effect of Geometrical Optical Aberrations

This function is known to be expressible in expanded form by

$$\Phi = \Phi^{(4)} + \Phi^{(6)} + \Phi^{(8)} + \ldots$$

where $\Phi^{(2k)}$ is a polynomial of degree 2k in the four coordinates X_o, Y_o, X', Y', containing these quantities only in combinations of powers of the three scalar invariants

$$\left(X_o^2 + Y_o^2\right), \; (X'^2 + Y'^2), \; (X_o X' + Y_o Y')$$

If diffraction is involved, the specific ray taken into consideration is diffracted at the above defined point $P'_o(X'_o, Y'_o, Z'_o)$ into all directions of the space behind the screen. If the diffraction effect is to be observed at point $P(X_1, Y_1, Z_1)$ in the neighborhood of P_1^*, the corresponding diffracted ray emerging from P'_o will intersect the exit pupil at $P'(X'_1, Y'_1, Z'_1)$. The intersection points of the ray $[P'P]$ with the actual wave front and the Gaussian reference sphere may now be defined by $\bar{Q}(\bar{X}', \bar{Y}', \bar{Z}')$ and $Q(X', Y', Z')$, respectively. The wave aberration again is given as the distance between these points:

$$\Phi = [\bar{Q}Q] = [P_o Q] - [P_o C] = \Phi(X_o, Y_o; X', Y')$$

The general expression obtained for the aberration function, therefore, remains unchanged.

According to the assumption of the smallness of the ratios between the exit pupil's linear dimensions and the radius of the Gaussian reference sphere, the coordinates X', Y' may be replaced by the coordinates X'_1, Y'_1 of the intersection point P'.

Instead of the real coordinates, Seidel's variables are introduced by the transformation

$$X_o = \frac{D_o x_o}{n_o \lambda_o} \qquad\qquad Y_o = \frac{D_o y_o}{n_o \lambda_o}$$

$$X_1 = -\frac{D_1 x_1}{n_1 \lambda_1} \qquad Y_1 = -\frac{D_1 y_1}{n_1 \lambda_1}$$

$$X'_o = \lambda_o \xi_o \qquad Y'_o = \lambda_o \eta_o$$

$$X'_1 = \lambda_1 \xi_1 \qquad Y'_1 = \lambda_1 \eta_1$$

where

D_o = the distance between the object and entrance pupil planes
D_1 = the distance between the entrance pupil and image planes
n_o = refractive index of the object space
n_1 = refractive index of the image space
λ_o = length unit in the entrance pupil plane
λ_1 = length unit in the exit pupil plane

Because, within the accuracy of Gaussian optics, one has

$$x_1 = x_o \; ; \; y_1 = y_o \; ; \; \xi_1 = \xi_o \; ; \; \eta_1 = \eta_o$$

one obtains the relation, with respect to the object and geometrical image points,

$$x_o = x_1^* \; ; \; y_o = y_1^*$$

For simplification, D_1 may be replaced by R, the refractive indices n_o and n_1 and the length unit λ_o may be taken equal to unity. Then, λ_1 characterizes the length unit in the exit pupil, as well as the magnification, $M' = \lambda_1/\lambda_o$ (dimensionless in this case, of course) between the two pupil planes.

The aberration function expressed in Seidel's variables takes the form

$$\Phi(x_1^*, y_1^*; \xi_1, \eta_1) = \sum_{k=2}^{\infty} \Phi^{(2k)}$$

where the polynomial $\Phi^{(2k)}$ is a linear power combination of degree $(2k)$ in the three basic quantities

$$r_o^2 = x_1^{*2} + y_1^{*2} \; ; \; \rho^2 = \xi_1^2 + \eta_1^2 \; ; \; \kappa^2 = x_1^* \xi_1 + y_1^* \eta_1$$

With respect to the primary or third-order aberration one has in usual notation

$$\Phi^{(4)} = -\frac{1}{4} B\rho^4 - C\kappa^4 - \frac{1}{2} Dr_o^2\rho^2 + Er_o^2\kappa^2 + F\rho^2\kappa^2$$

The function corresponding to the fifth-order aberrations contains the quantities $\rho^6, \kappa^6, r_o^2\rho^4, r_o^2\kappa^4, r_o^4\rho^2, r_o^4\kappa^2, \rho^4\kappa^2, \rho^2\kappa^4$; $\Phi^{(8)}$ the quantities $\rho^8, \kappa^8, r_o^2\rho^6$, $r_o^2\kappa^6, r_o^4\rho^4, r_o^4\kappa^4, r_o^6\rho^2, r_o^6\kappa^2, \rho^6\kappa^2, \rho^4\kappa^4, \rho^2\kappa^6$, and so on to infinity.

It is now recommended that Seidel's variables be related to the integration variables introduced above. This yields

$$x_1^* = -\lambda_1 X_1^*/R \qquad\qquad y_1^* = -\lambda_1 Y_1^*/R$$

$$\xi_1 \approx x/\lambda_1 = R\mu_o\xi/\lambda_1 \qquad\qquad \eta_1 \approx y/\lambda_1 = R\nu_o\eta/\lambda_1$$

By this transformation Seidel's third-order aberration function is expressible in the integration variables. The wave aberration may be simplified by arbitrarily choosing $X_1^* \equiv 0$. Thus, one obtains in explicit form

$$y_1^{*2} = \left(\frac{\lambda_1}{R}\right)^2 Y_1^{*2} = \text{constant}$$

$$\xi_1^2 + \eta_1^2 = \left(\frac{R}{\lambda_1}\right)^2 \left(\mu_o^2\xi^2 + \nu_o^2\eta^2\right)$$

$$y_1^*\eta_1 = -Y_1^*\nu_o\eta$$

and

$$\Phi^{(4)} = -\frac{1}{4} B\left(\frac{R}{\lambda_1}\right)^4 \left(\mu_o^4 \xi^4 + 2\mu_o^2 \nu_o^2 \xi^2 \eta^2 + \nu_o^4 \eta^4\right)$$

$$- CY_1^{*2} \nu_o^2 \eta^2$$

$$- \frac{1}{2} DY_1^{*2} \left(\mu_o^2 \xi^2 + \nu_o^2 \eta^2\right)$$

$$- E\left(\frac{\lambda_1}{R}\right)^2 Y_1^{*3} \nu_o \eta$$

$$- F\left(\frac{R}{\lambda_1}\right)^2 Y_1^{*}\left(\mu_o^2 \nu_o \xi^2 \eta + \nu_o^3 \eta^3\right)$$

The expressions for the aberration functions of higher order are obtained in the same way.

It is seen that the wave aberration consists of a linear combination of powers in ξ <u>and</u> η. For this reason, the exponential term in Fresnel-Kirchhoff's formula containing Φ in its exponent cannot be separated immediately into two terms containing only powers of ξ <u>or</u> η.

Section 4
EXPANSION OF THE INTEGRAND IN FRESNEL-KIRCHHOFF'S FORMULA

As mentioned in earlier sections of this report, the application of Zernike's circular polynomials enabled Nijboer to incorporate, in the case of diffraction by a circular aperture, the effect of aberrations by expanding the integrand in Fresnel-Kirchhoff's formula. This yielded an approximate, but nevertheless quite satisfactory, solution. Zernike's circle polynomials are complex functions in two real variables of the form

$$V_n^m(\rho \sin\theta\ ,\ \rho\cos\theta) = R_n^m(\rho)\, e^{jm\theta}$$

where $m \gtreqless 0$ and $n \geqq 0$ are integers, $n \geqq |m|$, and $n - |m|$ is even.

They satisfy the orthogonality relation

$$\int_0^{2\pi}\int_0^1 R_n^m(\rho)\, R_{n'}^{m'}(\rho)\, \exp\,[j(m - m')\theta]\, \rho\, d\rho\, d\theta = \frac{\pi}{n+1}\, \delta_n^{n'}\, \delta_m^{m'}$$

where δ_α^β is the well-known Kronecker symbol. This property is essential in Nijboer's integrand expansion because this expansion necessarily involves the occurrence of Zernike's radial polynomials $R_n^m(\rho)$ in all portions of the integral.

The rectangular aperture, unfortunately, does not have circular symmetry; rather, it is characterized by finite limits. Although these limits may be supposed to be symmetrical to the aperture center, they are different for each of the two variables, x and y. The only adequate polynomials having orthogonality properties in a finite integration interval are seen in Legendre's polynomials (this is also stated in a small note by

Zernike and Nijboer (Ref. 6) – "La théorie des images optiques," p. 228). To become orthogonal between finite limits (for example $\pm\mu_0$ or $\pm\nu_0$, respectively) the common expressions for these polynomials can easily be modified. However, it is believed more convenient to transform the variables x and y into an appropriate form such that the ordinary Legendre polynomials, with orthogonality property in the integration interval ± 1, can be used in expanding the terms of the diffraction integral. For this reason, in the previous section, the dimensionless integration variables, ξ and η, have been introduced.

The first step in the expansion to be performed refers to the exponential terms containing the first and second powers of ξ and η separately. The second step concerns the exponential term containing the wave aberration Φ.

According to Bauer (Ref. 7) one has

$$\exp(j\rho\cos\theta) = \sqrt{\frac{\pi}{2}}\sum_{s=0}^{\infty}(j)^s(2s+1)\frac{J_{s+1/2}(\rho)}{\rho^{1/2}}P_s(\cos\theta)$$

Substituting

$$\cos\theta = -\xi^2$$

and

$$\cos\theta = -\xi$$

and noting that

$$P_s(-\cos\theta) = (-1)^s P_s(\cos\theta)$$

one obtains

$$\exp\left(-j\,\frac{u}{2}\,\mu_o^2\xi^2\right) = \sqrt{\frac{\pi}{2}}\sum_{s=o}^{\infty}(-j)^s(2s+1)\,\frac{J_{s+1/2}\left(u\mu_o^2/2\right)}{\left(u\mu_o^2/2\right)^{1/2}}\,P_s(\xi^2)$$

$$\exp(-jv\mu_o\xi) = 2\sqrt{\frac{\pi}{2}}\sum_{s'=o}^{\infty}(-j)^{s'}\,\frac{2s'+1}{2}\,\frac{J_{s'+1/2}(v\mu_o)}{(v\mu_o)^{1/2}}\,P_{s'}(\xi)$$

and similar expressions for the exponential terms containing the variable η in their exponents.

The combined exponential term $\exp\left[-j\left(u\mu_o^2\xi^2/2 + v\mu_o\xi\right)\right]$ is expressible in the form of a double sum containing products $P_s(\xi^2)\,P_{s'}(\xi)$. The polynomials involved, unfortunately, are not orthogonal as can easily be shown.

However, the Legendre polynomials may be defined by

$$P_n(\xi) = \sum_{\nu=o}^{\frac{n-(1)}{2}}\frac{\beta_{n,\nu}}{\delta'_n}\,\xi^{n-2\nu} = 2^{-(n-1)}\sum_{\nu=o}^{\frac{n-(1)}{2}}(-1)^{\nu}\binom{2n-2\nu-1}{n-\nu-1}\binom{n-\nu}{\nu}\xi^{n-2\nu} \quad \text{for} \quad n \geqq 1$$

$$P_o(\xi) \equiv 1$$

where δ'_n is a common denominator for all coefficients. Similarly, the powers of ξ may be expressed by

$$\xi^n = \sum_{\nu=o}^{\frac{n-(1)}{2}}\frac{\alpha_{n,\nu}}{\delta_n}\,P_{n-2\nu}(\xi) = 2^n\sum_{\nu=o}^{\frac{n-(1)}{2}}2^{-2\nu}\,\frac{2n-4\nu+1}{n-2\nu+1}\left[\binom{n-\nu}{\nu}\Big/\binom{2n-2\nu+1}{n}\right]P_{n-2\nu}(\xi)$$

δ_n being again a common denominator for all coefficients.

From these definitions, it is obvious that by replacing, in $P_s(\xi)$, the variable ξ by ξ^2, i.e., by expressing $P_s(\xi^2)$ explicitly in terms of ξ^2, and by substituting, for the powers of ξ in the linear combination obtained, their expansions in terms of common Legendre polynomials in ξ, $P_s(\xi^2)$ is represented by a finite double sum in terms of those ordinary Legendre polynomials. Moreover, it is seen that the exponential term $\exp\left[-j\left(u\mu_o^2\xi^2/2 + v\mu_o\xi\right)\right]$ is expressible in the form of a quadruple sum in terms of products of common Legendre polynomials where every single term, of course, has the usual orthogonality property.

The exponential term $e^{jk\Phi}$, accordingly, is expanded into its power series in (jk^{Φ})

$$e^{jk\Phi} = 1 + j\frac{k}{1!}\Phi - \frac{k^2}{2!}\Phi^2 - j\frac{k^3}{3!}\Phi^3 + \frac{k^4}{4!}\Phi^4 + \ldots = \sum_{\ell=o}^{\infty}\frac{1}{\ell!}(jk\Phi)^2$$

In this expression, the wave aberration Φ, in general, can be considered as a double series in terms of power products $(\xi^{n-m}\eta^m)$ where $n = 0, 1, 2, \ldots \infty$, $m = 0, 1, 2, \ldots n$.

$$\Phi = \sum_{n=o}^{\infty}\sum_{m=o}^{n} K_{n-m,m}\,\xi^{n-m}\eta^m$$

where the constants $K_{n-m,m}$ are dependent on

- The off-axis position of the geometrical image point, X_1^* and Y_1^*
- The linear dimension parameters, μ_o and ν_o, of the aperture
- The radius R of the Gaussian reference sphere
- The unit length λ_1 of the exit pupil plane
- Certain aberration coefficients

Taking the ℓ^{th} power of the aberration function does not change the general expression, except that the coefficients of the expansion now are different. This may be expressed by

$$\Phi^{\ell} = \sum_{n=o}^{\infty} \sum_{m=o}^{n} K^{\ell}_{n-m\,,\,m}\, \xi^{n-m} \eta^{m}$$

where

$$\Phi^{o} \equiv 1$$

and yields

$$e^{jk\Phi} = \sum_{\ell=o}^{\infty} \frac{1}{\ell!} (jk)^{\ell} \sum_{n=o}^{\infty} \sum_{m=o}^{n} K^{\ell}_{n-m\,,\,m}\, \xi^{n-m} \eta^{m}$$

Evidently, in every single product of this expression, the powers of ξ and η can be expanded in terms of Legendre's polynomials.

By these expansions of the integrand terms, one can divide the two-dimensional diffraction integral into two one-dimensional integrals

$$\frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} \exp\left[j\left(k\Phi - \frac{u}{2}\,\mu_o^2\xi^2 - v\mu_o\xi - \frac{u}{2}\,\nu_o^2\eta^2 - w\nu_o\eta\right)\right] d\xi\, d\eta$$

$$= \sum_{\ell=o}^{\infty} \frac{1}{\ell!} (jk)^{\ell} \sum_{n=o}^{\infty} \sum_{m=o}^{n} K^{\ell}_{n-m\,,\,m} V^{n-m}_{\ell}(u\,,\,v)\, W^{m}_{\ell}(u\,,\,w)$$

Then, the disturbance at the observation point P is given by

$$U(u,v,w) = -j\,e^{ju}\,\frac{A}{\lambda}\,4\mu_o\nu_o \sum_{\ell=o}^{\infty} \frac{1}{\ell!}(jk)^{\ell} \sum_{n=o}^{\infty} \sum_{m=o}^{n} K^{\ell}_{n-m,\,m} V_{\ell}^{n-m}(u,v)\, W_{\ell}^{m}(u,w)$$

where

$$V_{\ell}^{n-m}(u,v) = \frac{\pi}{2} \sum_{s=o}^{\infty} \sum_{s'=o}^{\infty} (-j)^{s+s'} (2s+1) \frac{J_{s+1/2}\left(u\mu_o^2/2\right)}{\left(u\mu_o^2/2\right)^{1/2}} \frac{2s'+1}{2} \frac{J_{s'+1/2}(v\mu_o)}{(v\mu_o)^{1/2}} \int_{-1}^{1} \xi^{n-m} P_s(\xi^2) P_{s'}(\xi)\, d\xi$$

$$W_{\ell}^{m}(u,w) = \frac{\pi}{2} \sum_{t=o}^{\infty} \sum_{t'=o}^{\infty} (-j)^{t+t'} (2t+1) \frac{J_{t+1/2}\left(u\nu_o^2/2\right)}{\left(u\nu_o^2/2\right)^{1/2}} \frac{2t'+1}{2} \frac{J_{t'+1/2}(w\nu_o)}{(w\nu_o)^{1/2}} \int_{-1}^{1} \eta^{m} P_t(\eta^2) P_{t'}(\eta)\, d\eta$$

Section 5

EVALUATION OF THE DIFFRACTION INTEGRAL

After the reduction of Fresnel-Kirchhoff's diffraction formula for the rectangular aperture to a much simpler form, the remaining integrations to be performed concern only integrals of the type

$$\int_{-1}^{1} x^k P_r(x^2) P_{r'}(x)\, dx$$

where x stays for any of the variables ξ and η; r, r' replace any of the subscripts s, s' and t, t', respectively; and k replaces any of the exponents $(n-m)$ and m. In order to evaluate this integral, obviously one will have to make use of the orthogonality property of Legendre's polynomials

$$\int_{-1}^{1} P_\alpha(x) P_\beta(x)\, dx = \frac{2}{2\beta + 1}\, \delta_\alpha^\beta$$

δ_α^β being the Kronecker symbol.

It is deemed advisable to introduce new polynomials defined by

$$\boldsymbol{\rho}_r^k(x) = x^k P_r(x^2) = \sum_{\rho=o}^{\frac{r-(1)}{2}} \frac{\beta_{r,\rho}}{\delta_r^!} x^{2r+k-4\rho} = 2^{-(r-1)} \sum_{\rho=o}^{\frac{r-(1)}{2}} (-1)^\rho \binom{2r-2\rho-1}{r-\rho-1} \binom{r-\rho}{\rho} x^{2r+k-4\rho}$$

for $r \geqq 1$

$$\boldsymbol{\rho}_o^k(x) = x^k$$

Expressing the powers of x by their expansions in terms of Legendre polynomials,

$$x^{2r+k-4\rho} = \sum_{\kappa=0}^{\frac{2r+k-4\rho-(1)}{2}} \frac{\alpha_{2r+k-4\rho,\kappa}}{\delta_{2r+k-4\rho}} P_{2r+k-4\rho-2\kappa}(x)$$

$$= 2^{2r+k-4\rho} \sum_{\kappa=0}^{\frac{2r+k-4\rho-(1)}{2}} 2^{-2\kappa} \frac{4r+2k-8\rho-4\kappa+1}{2r+k-4\rho-2\kappa+1}$$

$$\times \left[\binom{2r+k-4\rho-\kappa}{\kappa} \Big/ \binom{4r+2k-8\rho-2\kappa+1}{2r+k-4\rho}\right] P_{2r+k-4\rho-2\kappa}(x)$$

$$x^{k} = \sum_{\kappa=0}^{\frac{k-(1)}{2}} \frac{\alpha_{k,\kappa}}{\delta_k} P_{k-2\kappa}(x)$$

one obtains

$$\rho_r^k(x) = \sum_{\rho=0}^{\frac{r-(1)}{2}} \frac{\beta_{r,\rho}}{\delta'_r} \sum_{\kappa=0}^{\frac{2r+k-4\rho(-1)}{2}} \frac{\alpha_{2r+k-4\rho,\kappa}}{\delta_{2r+k-4\rho}} P_{2r+k-4\rho-2\kappa}(x)$$

$$\rho_o^k(x) = \sum_{\kappa=0}^{\frac{k-(1)}{2}} \frac{\alpha_{k,\kappa}}{\delta_k} P_{k-2\kappa}(x)$$

The values of $\alpha_i, \beta_i, \delta_i, \delta'_i$ are known from Salzer's (Ref. 8) tables of the powers of x in terms of Legendre polynomials (up to order twenty-four) and from Tallquist's (Ref. 9) tables of the Legendre polynomials (up to order sixteen) and can easily be extended to higher orders.

The expression obtained for the polynomials $\rho^k_r(x)$ may be more simplified by taking into account the fact that some of the coefficients of the expansion are to be related to a Legendre polynomial of the same order. This yields

$$\rho^k_r(x) = \sum_{i=o}^{\frac{2r+k-(1)}{2}} \frac{N^k_{r,i}}{D^k_r} P_{2r+k-2i}(x)$$

where D^k_r is a common denominator for all the coefficients. The coefficients $N^k_{r,i}$ and D^k_r have been computed for $0 \leqq r \leqq 12$ and $0 \leqq k \leqq 24$ with the restriction $[2r + k - (1)]/2 \leqq 12$, and are available by request (Ref. 10).

By this procedure, the evaluation of the typical integral is reduced to

$$\int_{-1}^{1} x^k P_r(x^2) P_{r'}(x)\, dx = \int_{-1}^{1} \rho^k_r(x) P_{r'}(x)\, dx$$

$$= \sum_{i=o}^{\frac{2r+k-(1)}{2}} \frac{N^k_{r,i}}{D^k_r} \int_{-1}^{1} P_{2r+k-2i}(x) P_{r'}(x)\, dx$$

$$= \sum_{i=o}^{\frac{2r+k-(1)}{2}} \frac{N^k_{r,i}}{D^k_r} \frac{2}{2r' + 1}$$

where necessarily $r' = 2r + k - 2i$.

For convenience the functions are defined

$$G_r^k(z) = (-j)^{r'-k}\,\frac{2r'+1}{2}\,\frac{J_{r'+1/2}(z)}{z^{1/2}}\int_{-1}^{1}\rho_r^k(x)P_{r'}(x)\,dx = \sum_{i=o}^{\frac{2r+k-(1)}{2}}(-1)^{r-i}\,\frac{N_{r,i}^k}{D_r^k}\,\frac{J_{2r+k-2i+1/2}(z)}{z^{1/2}}$$

where z stays for any of the variables $(v\mu_o)$ and $(w\nu_o)$.

Substituting these results into the expressions for V_ℓ^{n-m} and W_ℓ^m obtained above, one has finally

$$V_\ell^{n-m}(u,v) = (-j)^{n-m}\,\frac{\pi}{2}\sum_{s=o}^{\infty}(-j)^s(2s+1)\,\frac{J_{s+1/2}\left(u\mu_o^2/2\right)}{\left(u\mu_o^2\;2\right)^{1/2}}\,G_s^{n-m}(v\mu_o)$$

$$W_\ell^m(u,w) = (-j)^m\,\frac{\pi}{2}\sum_{t=o}^{\infty}(-j)^t(2t+1)\,\frac{J_{t+1/2}\left(u\nu_o^2/2\right)}{\left(u\nu_o^2/2\right)^{1/2}}\,G_t^m(w\nu_o)$$

where

$$G_s^{n-m}(v\mu_o) = \sum_{\sigma=o}^{\frac{2s+n-m-(1)}{2}}(-1)^{s-\sigma}\,\frac{N_{s,\sigma}^{n-m}}{D_s^{n-m}}\,\frac{J_{2s+n-m-2\sigma+1/2}(v\mu_o)}{(v\mu_o)^{1/2}}$$

$$G_t^m(w\nu_o) = \sum_{\tau=o}^{\frac{2t+m-(1)}{2}}(-1)^{t-\tau}\,\frac{N_{t,\tau}^m}{D_t^m}\,\frac{J_{2t+m-2\tau+1/2}(w\nu_o)}{(w\nu_o)^{1/2}}$$

Normally, the three-dimensional intensity distribution is now obtained as

$$I(u,v,w) = |U(u,v,w)|^2 = I_o \left| \sum_{\ell=o}^{\infty} \frac{1}{\ell!} (jk)^{\ell} \sum_{n=o}^{\infty} \sum_{m=o}^{n} K^{\ell}_{n-m,m} V_{\ell}^{n-m}(u,v) W_{\ell}^{m}(u,w) \right|^2$$

where

$$I_o = \left(\frac{A}{\lambda} 4\mu_o \nu_o\right)^2$$

represents the intensity at the center of the aberration-free diffraction pattern.

The solution in the Gaussian image plane is of special interest. Because of

$$\lim_{\zeta \to o} \frac{J_{r+1/2}(\zeta)}{\zeta^{1/2}} = \sqrt{\frac{2}{\pi}}\, \delta_r^o$$

one obtains

$$V_{\ell}^{n-m}(o,v) = (-j)^{n-m} \sqrt{\frac{\pi}{2}}\, G_o^{n-m}(v\mu_o) = (-j)^{n-m} \sqrt{\frac{\pi}{2}} \sum_{\sigma=o}^{\frac{n-m-(1)}{2}} (-1)^{\sigma} \frac{N^{n-m}_{o,\sigma}}{D_o^{n-m}} \frac{J_{n-m-2\sigma+1/2}(v\mu_o)}{(v\mu_o)^{1/2}}$$

$$W_{\ell}^{m}(o,w) = (-j)^{m} \sqrt{\frac{\pi}{2}} \sum_{\tau=o}^{\frac{m-(1)}{2}} (-1)^{\tau} \frac{N^{m}_{o,\tau}}{D_o^{m}} \frac{J_{m-2\tau+1/2}(w\nu_o)}{(w\nu_o)^{1/2}}$$

Hence

$$I(o,v,w) = I_o \left| \sum_{\ell=o}^{\infty} \frac{1}{\ell!} (jk)^{\ell} \sum_{n=o}^{\infty} \sum_{m=o}^{n} K^{\ell}_{n-m,m} (-j)^n \frac{\pi}{2} \right.$$

$$\left. \times \sum_{\sigma=o}^{\frac{n-m-(1)}{2}} (-1)^{\sigma} \frac{N^{n-m}_{o,\sigma}}{D_o^{n-m}} \frac{J_{n-m-2\sigma+1/2}(v\mu_o)}{(v\mu_o)^{1/2}} \sum_{\tau=o}^{\frac{m-(1)}{2}} (-1)^{\tau} \frac{N^{m}_{o,\tau}}{D_o^{m}} \frac{J_{m-2\tau+1/2}(w\nu_o)}{(w\nu_o)^{1/2}} \right|^2$$

This yields for the aberration-free optical system, characterized by $n = m \equiv 0$,

$$I(o,v,w) = I_o \left| \frac{\pi}{2} \frac{J_{1/2}(v\mu_o)}{(v\mu_o)^{1/2}} \frac{J_{1/2}(w\nu_o)}{(w\nu_o)^{1/2}} \right|^2 = I_o \left(\frac{\sin v\mu_o}{v\mu_o} \right)^2 \left(\frac{\sin w\nu_o}{w\nu_o} \right)^2$$

i.e., the classical expression for the diffraction pattern due to a rectangular aperture.

Furthermore, the three-dimensional diffraction pattern intensity distribution near the focus of an aberration-free optical system is also believed to be of interest. This distribution is represented by:

$$I(u,v,w) = I_o \left| V_o^o(u,v) \right|^2 \left| W_o^o(u,w) \right|^2$$

where

$$V_o^o(u,v) = \frac{\pi}{2} \sum_{s=o}^{\infty} (-j)^s (2s+1) \frac{J_{s+1/2}\left(u\mu_o^2/2\right)}{\left(u\mu_o^2/2\right)^{1/2}} G_s^o(v,\mu_o)$$

$$W_o^o(u,w) = \frac{\pi}{2} \sum_{t=o}^{\infty} (-j)^t (2t+1) \frac{J_{t+1/2}\left(u\nu_o^2/2\right)}{\left(u\nu_o^2/2\right)^{1/2}} G_t^o(w,\nu_o)$$

$$G_s^o(v\mu_o) = \sum_{\sigma=o}^{s} (-1)^{s-\sigma} \frac{N_{s,\sigma}^o}{D_s^o} \frac{J_{2s-2\sigma+1/2}(v\mu_o)}{(v\mu_o)^{1/2}}$$

$$G_t^o(w\nu_o) = \sum_{\tau=o}^{t} (-1)^{t-\tau} \frac{N_{t,\tau}^o}{D_t^o} \frac{J_{2t-2\tau+1/2}(w\nu_o)}{(w\nu_o)^{1/2}}$$

This equation, of course, yields the above-derived, classical expression for the diffraction pattern of a rectangular aperture when taking $u = 0$, i.e., $s = t \equiv 0$.

Section 6
SUMMARY

The Fresnel-Kirchhoff integral at an arbitrary observation point in the neighborhood of the geometrical image of a specific object point represents the disturbance due to the diffraction by a rectangular aperture in the presence of geometrical aberrations. In this paper it is shown that this integral can be expanded into a series. The expansion is based primarily on the application of Bauer's formula containing Bessel functions of the order $(n + 1/2)$ where n is an integer and also on the representation of the exponential term containing the wave aberration in its exponent by a power series. By this procedure, a significant simplification is obtained which is seen in the conversion of the two-dimensional diffraction integral into a summation. The single terms are divided into products of two functions where only one-dimensional integrations are to be performed.

To obtain a general solution of the diffraction integral, new polynomials are introduced and expanded in terms of Legendre polynomials. Applying the orthogonality property of Legendre's polynomials yields the definition of special functions, containing essentially linear combinations of Bessel functions of the order $(n + 1/2)$.

By these special functions, the disturbance and the three-dimensional intensity distribution about the arbitrary observation point in the neighborhood of the geometrical image point are expressible in a general form.

Particular solutions are obtained for the two-dimensional intensity distribution of the diffraction pattern in the Gaussian image plane in the presence of aberrations and for the three-dimensional intensity distribution near the focus of an aberration-free optical system.

Section 7
REFERENCES

CITED REFERENCES

1. M. Born and E. Wolf, Principles of Optics, Pergamon Press, New York, 1959, p. 374, also pp. 76 and 202.
2. E. C. J. v. Lommel, Abh. d. Bayerischen Akademie d. Wissenschaften, Vol. 15, München, Abt. 2, 1885, p. 233; Abt. 3, 1886, p. 531
3. E. Wolf, Reports on Progress in Optics, Vol. 14, Physical Society, London, 1951, p. 95
4. B. R. A. Nijboer, Thesis, University of Groningen, 1942 Physica, Vol. 10, 1943, p. 679; Vol. 13, 1947, p. 605,
5. F. Zernike, Physica, Vol. 1, 1934, p. 689
6. F. Zernike and B. R. A. Nijboer, "La théorie des images optiques," Editions de la Revue d'Optique, Paris, 1949, p. 227
7. Bauer, J. für Mathematik, Vol. 56, 1859, p. 104
8. H. E. Salzer, Math. Tables and Other Aids for Computation, Vol. 3, 1948/49, p. 16
9. Hj. Tallquist, Act. Soc. Fennicae, Nova Series A, Vol. 2, No. 4, 1937, p. 1
10. H. P. Greinel, Functions and Tables of Coefficients Needed for the Description of the Diffraction Pattern, in the Presence of Aberrations, Due to a Stationary Rectangular Aperture, Internal Report, Electronic Sciences Lab., Lockheed Missiles & Space Company, Palo Alto, Calif.

NOT SPECIFICALLY CITED REFERENCES

1.1. H. v. Helmholtz, J. für Mathematik, Vol. 57, 1859, p. 7

1.2. G. Kirchhoff, Berliner Berichte, 1882, p. 641; Annalen der Physik, (2), Vol. 18, 1883, p. 663

1.3. H. A. Buchdahl, Optical Aberration Coefficients, Oxford Univ. Press, 1954

1.4. M. Herzberger, Modern Geometrical Optics, New York, Interscience, 1958

1.5. L. Seidel, Astronomische Nachrichten, Vol. 43, No. 1027, 1856, p. 289; No. 1028, p. 305; No. 1029, p. 321

1.6. K. Schwarzschild, Abh. d. Koeniglichen Gesellschaft d. Wissenschaften, Goettingen, Math.-phys. Klasse, Vol. 4, No. 1, 2, and 3, 1905/1906

1.7. A. Kohlschuetter, Dissertation, Goettingen, 1908

1.8. G. C. Steward, Trans. of the Cambridge Phil. Soc., Vol. 23, 1926, p. 235

1.9. M. Herzberger, J. Optical Soc. Am., Vol. 29, 1939, p. 395

1.10. F. Wachendorf, Optik, Vol. 5, 1949, p. 80

1.11. J. Focke, Dissertation, Leipzig, 1951

1.12. N. Chako, Trans. Chalmers Univ. of Technology, Gothenburg, 1957, p. 191

1.13. H. A. Buchdahl, J. Optical Soc. Am., Vol. 46, 1956, p. 941; Vol. 48, 1958, pp. 563, 747, 757; Vol. 49, 1959, p. 1113; Vol. 50, 1960, pp. 534, 539, 678 Vol. 51, 1961, p. 608

4.1. K. Nienhuis, Thesis, University of Groningen, 1948

4.2. K. Nienhuis and B. R. A. Nijboer, Physica, Vol. 14, 1948, p. 590

5.1. F. Zernike and H. C. Brinkman, Proc. of Section of Sciences, Akademie van Wetenschappen, Amsterdam, Vol. 38, 1935, p. 161

5.2. A. B. Bhatia and E. Wolf, Proc. of the Cambridge Phil. Soc., Vol. 50, 1954, p. 40

8.1. G. N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge Univ. Press, 1958, p. 128